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Coaction crossed products of Hilbert C^*-bimodules by finite groups

Tsuyoshi KAJIWARA*

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Abstract

In this paper, we define coaction crossed product of Hilbert C*-bimodule by finite groups. We show that resulting bimodule is of finite type, and compute indices of them. We present a Takesaki-Takai duality theorem which is some non-commutative generalization of abelian groups case in [KW2].

Keywords: Hilbert bimodule, Coaction, Crossed product.

1. Introduction

In [KW1], we defined Hilbert C*-bimodules of finite type, and showed fundamental matters and examples. After this, in [KW2], we construct crossed products of Hilbert C*-bimodules of finite type by countable discrete groups. We showed that the crossed product bimodules are also Hilbert C*-bimodules of finite type, showed some categorical formulas, and showed Takesaki-Takai duality theorem for finite abelian groups. In [KW3], we made a generalization to crossed products by bundles. In [K], we considered crossed product bimodules by locally compact non-discrete groups using the theory of countably generated Hilbert C*-bimodules presented in [KPW].

In this paper, we construct a coaction crossed product bimodule by finite group. Then, using the theory of linking algebra in [BGR], we may define actions and inner products on crossed products bimodules. Moreover, we construct finite basis in two sides, and we show that this crossed product bimodules are of finite type. At last we show Takesaki-Takai duality theorem, which is a generalization of that in [KW2].

2. Hilbert C*-bimodule of finite type

We review an equivalent definition of Hilbert C*-bimodules of finite type following [KW3]. Let A and B be unital C*-algebras. Let X be a C-vector space.

Definition 1. ([KW1]) X is called a Hilbert A-B bimodule of finite type if the followings hold.

1. X is a left A-module.
2. X has a left self adjoint A-inner product ${}_A(\cdot|\cdot)$. The linear span of the range of this inner product is equal to A.
3. ${}_A(ax|y) = a\,{}_A(x|y)$.
4. X is a right B-module.
5. X has a right self adjoint (not necessarily positive) B-inner product $(\cdot|\cdot)_B$. The linear span of the range of this inner product is equal to B.

*Department of Environmental and Mathematical Sciences, Faculty of Environmental Science and Technology, Okayama University, Okayama, 700 Japan.

6. $(x|yb)_B = (x|y)_B b$.
7. Left A action and right B action commute each other.
8. $(ax|y)_B = (x|a^*y)_B$ and ${}_A(x|yb) = {}_A(xb^*|y)$.
9. There exists a finite subset $\{u_i\}_i$ in X such that $\sum_i u_i(u_i|x)_B = x$ for all $x \in X$.
10. There exists a finite subset $\{v_j\}_j$ in X such that $\sum_j {}_A(x|v_j)v_j = x$ for all $x \in X$.

We call the subset $\{u_i\}_i$ the right B-basis of X and $\{v_j\}_j$ the left A-basis of X.

([KW1]) If X satisfies all conditions in Definition 1, the followings hold.

1. ${}_A(x|x) \geq 0$ and $(x|x)_B \geq 0$. ${}_A(x|x) = 0$ if and only if $x = 0$ and $(x|x)_B = 0$ if and only if $x = 0$.
2. $\|{}_A(xb|xb)\| \leq \|b\|^2\|{}_A(x|x)\|$ and $\|(ax|ax)_B\| \leq \|a\|^2\|(x|x)_B\|$.
3. $\|{}_A(x|x)\| \leq \|\sum_i {}_A(u_i|u_i)\|\|(x|x)_B\|$ and
   $\|(x|x)_B\| \leq \|\sum_j (v_j|v_j)_B\|\|{}_A(x|x)\|$ In particular, two norms
   $\|x\|_A = \|(x|x)\|^{1/2}$ and $\|x\|_B = \|(x|x)_B\|^{1/2}$ on X are equivalent.
4. X is complete with respect to $\|\cdot\|_A$ (equivalently $\|\cdot\|_B$).

3. Coaction Crossed products by finite groups

We review the definition of coaction C*-crossed products by finite groups. Let A be a unital C*-algebra, and G be a finite group. Let Δ be a *-homomorphism for $C^*(G)$ to $C^*(G) \otimes C^*(G)$ given by $\Delta(\lambda(g)) = \lambda(g) \otimes \lambda(g)$ where $\lambda(g)$ is defined by $\lambda(g)\xi(g') = \xi(g^{-1}g')$ for $\xi \in l^2(G)$.

Definition 2. A unital *-homomorphism δ from A to $A \otimes C^*(G)$ is called a coaction of G on A if the following holds.

$$(\delta \otimes I_{C^*(G)}) \circ \delta = (I_A \otimes \Delta) \circ \delta$$

Since G is a finite group, all tensor products are algebraic and no problem concerning C*-tensor products occur.

For $f \in C(G)$, we define the multiplication operator M_f in $B(l^2(G))$ by $M_f(\xi)(g) = f(g)\xi(g)$ for $\xi \in l^2(G)$.

Definition 3. Let A be represented on a Hilbert space H faithfully. The coaction crossed product $A \times_\delta \hat{G}$ of A by G is a C*-algebra generated by $\{\delta(a), I \otimes M_f | a \in A, f \in C(G)\}$ in $B(H \otimes l^2(G))$. The coaction crossed product C*-algebra is independent of the choice of the representation space H of A.

Following Lemma is a coaction version of Fourier expansion and is well known. We present proof because we do not know reference. Let π be an irreducible unitary representation of G, H_π be a representation space of π, and d_π be the dimension of H_π. We fix ortho normal basis $\{v_i^\pi\}_{i=1,\ldots,d_\pi}$. We put $e_{i,j}^\pi(g) = (\pi(g)v_j^\pi, v_i^\pi)$ for $g \in G$. We enumerate $\{e_{i,j}^\pi\}$ and put this finite set $\{\omega_p\}_{p=1,\cdots,n}$ where n is the order of G.

Lemma 4. Each element of the coaction crossed product $A \times_\delta \hat{G}$ may be expressed of the form

$$\sum_{p=1}^{n} \delta(a_p)(I \otimes M(\omega_p))$$

This expression is unique. Another expression

$$\sum_{p=1}^{n}(I \otimes M(\omega_p))\delta(a'_p)$$

is also possible and unique.

Proof Since there exists a conditional expectation E from $A \times_{\delta_A} \hat{G}$ to A by the action dual to δ. Then, for example by [S], $\{I \otimes M(\omega_i)\}_i$ is a Pimsner-Popa basis for E, and the above expression is possible. Let $f_g(g') = \delta_{g,g'}$. Then $\{f_g\}_{g \in G}$ is a base in $C(G)$, and the expression $\sum_{g \in G} \delta(a_g)(I \otimes M_{f_g})$ is unique. Then the above expression using $\{\omega_p\}_p$ is also unique. □

Let A and B be unital C*-algebras and X be a Hilbert A-B bimodule of finite type. Suppose that there exist a coaction δ_A of G on A, coaction δ_B of G on B and a bounded linear map δ_X from X to $X \otimes \mathrm{C}^*(G)$ such that $(\delta_X \otimes I_{\mathrm{C}^*(G)})\delta_X(x) = (I_X \otimes \Delta)\delta_X(x)$ for $x \in X$. We remark that $X \otimes \mathrm{C}^*(G)$ is an $A \otimes \mathrm{C}^*(G)$-$B \otimes \mathrm{C}^*(G)$ bimodule of finite type, which is an example of outer tensor product [KW1].

Definition 5. The system $(X, A, B, \delta_X, \delta_A, \delta_B, G)$ is called a G equivariant co-system if the following holds.

1. $\delta_X(ax) = \delta_A(a)\delta_X(x), \quad \delta_X(xb) = \delta_X(x)\delta_B(b)$ for $a \in A$, $b \in B$ and $x \in X$.
2. $\delta_A({}_A(x|y)) = {}_{A\otimes \mathrm{C}^*(G)}(\delta_X(x)|\delta_X(y), \quad \delta_B((x|y)_B) = (\delta_X(x)|\delta_X(y))_{B\otimes \mathrm{C}^*(G)}$.

We denote X^r as the imprimitivity $K_B(X_B)$-A bimodule changing left action and left inner product from X. We denote C^r as the linking algebra of $(K_B(X_B), X, A)$ ([BGR]).

Lemma 6. Let $(X, A, B, \delta_X, \delta_A, \delta_B, G)$ be a G equivariant co-system. Then there exists a coaction δ_{C^r} of G on C^r canonically.

Proof By Proposition 2.5 in [Bu], we may define a coaction δ_K of G on $\mathbf{K}_B(X_B)$ such that $\delta_X(Tx) = \delta_K(T)\delta_X(x)$ for $T \in \mathbf{K}_B(X_B)$ and $x \in X$. If $T = \theta^r_{x,y}$, we have $\delta_K(\theta^r_{x,y}) = \theta^r_{\delta_X(x),\delta_X(y)}$. Let $\overline{\delta_X}$ be the linear map canonically defined on $\overline{X}$ from δ_X. We may define a action δ_{C^r} of G on C^r as follows.

$$\delta_{C_r}\left(\begin{bmatrix} a & x \\ x' & k \end{bmatrix}\right) = \begin{bmatrix} \delta_A(a) & \delta_X(x) \\ \overline{\delta_X}(x') & \delta_K(k) \end{bmatrix}$$

for $a \in A$, $x \in X$, $x' \in \overline{X}$ and $k \in \mathbf{K}_B(X_B)$. □

We make a coaction crossed product of $C^r \times_{\delta_{C^r}} \hat{G}$. By Lemma 4, elements of $C^r \times_{\delta_{C^r}} \hat{G}$ may be expressed of the form $\sum_p \delta_{C^r}(c_p)(I \otimes M(\omega_p))$ and this expression is unique. Then by [BGR], $(1,2)$ part of $C^r \times_{\delta_{C^r}} \hat{G}$ is the imprimitivity bimodule between $A \times_{\delta_A} \hat{G}$ and $B \times_{\delta_B} \hat{G}$.

Definition 7. We denote $(1,2)$ part of $C^r \times_{\delta_{C^r}} \hat{G}$ by $X^r \times_{\delta_X} \hat{G}$.

Similarly, we define C^l, δ_{C^l} and $X^l \times_{\delta_X} \hat{G}$.

Lemma 8. $X^r \times_{\delta_X} \hat{G}$ is a right Hilbert $B \times_{\delta_B} \hat{G}$ module and $X^l \times_{\delta_X} \hat{G}$ is a left Hilbert $A \times_{\delta_A} \hat{G}$ module.

Lemma 9. $X^r \times_{\delta_X} \hat{G}$ and $X^l \times_{\delta_X} \hat{G}$ are naturally isomorphic as linear spaces.

Proof Elements in $X^r \times_{\delta_X} \hat{G}$ and $X^l \times_{\delta_X} \hat{G}$ have the same expression because they are (2,1) component of coaction C*-crossed products. Since Fourier expansion is unique, we may define a linear isomorphism canonically. □

By these lemma, $C^r \times_{\delta_{C^r}} \hat{G}$ may be considered as left Hilbert $A \times_{\delta_A} \hat{G}$ module.

Let $\{u_i\}_i$ be a finite base of X_B and $\{v_j\}_j$ be a finite left base of ${}_AX$. We put $\hat{u}_i = \delta_X(u_i)$ and $\hat{v_j} = \delta_X(v_j)$.

□

Lemma 10. $\{\hat{u}_i\}_i$ is a right $B \times_{\delta_B} \hat{G}$ basis, and $\{\hat{v}_j\}_j$ is a left $A \times_{\delta_A} \hat{G}$ base.

Proof We only consider right base case. In $C^r \times_{\delta_{C^r}} \hat{G}$, inner products and actions are expressed by products in C*-algebras. Then for $x = \sum_p \delta_X(x_p)(I \otimes M_{\omega_p}) \in X^r \times_{\delta_X} \hat{G}$,

$$\begin{aligned}
\sum_j \hat{v}_j(\hat{v}_j|x)_{B\times_{\delta_B}\hat{G}} &= \sum_j \delta_X(v_j)\delta_X(v_j)^* \sum_p \delta_X(x_p)(I \otimes M_{\omega_p}) \\
&= \sum_p \delta_X(\sum_j v_j(v_j|x_p)_B)(I \otimes M_{\omega_p}) \\
&= \sum_p \delta_X(x_p)(I \otimes M_{\omega_p}) \\
&= x
\end{aligned}$$

□

The conditional expectation E_0 from $\mathbf{K}_B(X_B)$ to A satisfies $E_0(\delta_K(T)) = \delta_A(E_0(T))$. Then the restriction of δ_K to A is the original δ_A. Then $A \times_{\delta_A} \hat{G}$ can be considered as the C*-subalgebra of $\mathbf{K}_B(X_B) \times_{\delta_K} \hat{G}$. We may consider two left actions of $A \times_{\delta_A} \hat{G}$ on $X^r \times_{\delta_X} \hat{G}$ by the inclusion $A \times_{\delta_A} \hat{G} \subset C^l \times_{\delta^l_C} \hat{G}$ and the inclusion of $A \times_{\delta_A} \hat{G} \subset \mathbf{K}_B(X_B) \times_{\delta_K} \hat{G}$.

Lemma 11. Then left actions as above of $A \times_{\delta_A} \hat{G}$ on $X^r \times_{\delta_X} \hat{G}$ are identical.

Proof Let $x = \sum_p \delta_X(x_p)(I \otimes M_{\omega_p})$. Then we have,

$$\begin{aligned}
\delta_A(a)\sum_p \delta_X(x_p)(I \otimes M_{\omega_p}) &= \sum_p \delta_X(ax_p)(I \otimes M_{\omega_p}) \\
(I \otimes M_\omega)\sum_p \delta_X(x_p)(I \otimes M_{\omega_p}) &= \sum_p\sum_q E\left((I \otimes M_\omega)\delta_X(x_p)(I \otimes M_{\overline{\omega_q}})\right)(I \otimes M_{\omega_q})(I \otimes M_{\omega_p})
\end{aligned}$$

Then $E\left((I \otimes M_\omega)\delta_X(x_p)(I \otimes M_{\overline{\omega_q}})\right)$ is independent of the representation spaces of X. □

By these arguments, we have the following theorem.

Theorem Let A and B be unital C*-algebras, X be a Hilbert A-B bimodule of finite type and G be a finite group. Let $(X, A, B, \delta_X, \delta_A, \delta_B, G)$ be a G equivariant co-system. Then $X^r \times_{\delta_X} \hat{G}$ is made into a Hilbert $A \times_{\delta_A} \hat{G}$-$B \times_{\delta_B} \hat{G}$ bimodule and is of finite type.

Definition 12. We call this bimodule $X \times_{\delta_X} \hat{G}$.

We may compute indices of the coaction crossed products.

Proposition 13. We have $\text{rind}[X \times_{\delta_X} \hat{G}] = \delta_A(\text{rind}[X])$ and $\text{lind}[X \times_{\delta_X} \hat{G}] = \delta_B(\text{lind}[X])$.

Proof We only show the case of rind. We have

$$\begin{aligned}
\sum_i {}_{A\times_{\delta_A}\hat{G}}(\hat{u}_i|\hat{u}_i) &= \sum_i {}_{A\times_{\delta_A}\hat{G}}(\delta_X(u_i)|\delta_X(u_i)) \\
&= \sum_i \delta_A({}_A(u_i|u_i)) \\
&= \delta_A(\text{rind}[X])
\end{aligned}$$

□

Let A and B be unital C*-algebras, and X a Hilbert A-B bimodule of finite type, G a finite group, and $(X, A, B, \gamma, \alpha, \beta, , G)$ be a G-equivariant system ([KW2]). The by [KW2], $X \times_\gamma G$ is defined and is made into a $A \times_\alpha G$-$B \times_\beta G$ bimodule of finite type.

There exists coactions δ_A, δ_B and δ_X of G of $A \times_\alpha G$, $B \times_\beta G$ and $X \times_\gamma G$ such that

$$\delta_A(a\lambda_g) = a\lambda_g \otimes \lambda_g, \quad \delta_B(b\lambda_g) = b\lambda_g \otimes \lambda_g, \quad \delta_X(x\lambda_g) = x\lambda_g \otimes \lambda_g$$

Then $(X \times_\gamma G, A \times_\alpha G, B \times_\beta, \delta_X, \delta_A, \delta_B, G)$ is a G-equivariant co-system.

On the other hand, if $(X, A, B, \delta_X, \delta_A, \delta_B, G)$ is a G-equivariant co-system. Then there exist actions $\hat{\delta}_X$, $\hat{\delta}_A$ and $\hat{\delta}_B$ dual to δ_X, δ_A and δ_B and $(X \times_{\delta_X} \hat{G}, A \times_{\delta_A} \hat{G}, B \times_{\delta_B} \hat{G}, \hat{\delta}_X, \hat{\delta}_A, \hat{\delta}_B, G)$ is a G-equivariant system.

Proposition 14. (Takesaki-Takai duality Theorem) $(X \times_\gamma G) \times_{\delta_X} \hat{G} \simeq X \otimes \mathbf{C}(l^2(G))$, and $(X \times_{\delta_X} \hat{G}) \times_{\hat{\delta_X}} G \simeq X \otimes \mathbf{C}(l^2(G))$ hold and the above isomorphisms are compatible with Takesaki-Takai dualities of C*-crossed products.

Proof The proof is similar to [KW2] and [K], and use crossed products of linking algebras. □

Remark 15. We may construct crossed products bimodule by finite dimensional Kac C*-algebras in a similar way.

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